7 Algebra: Graphs, Functions and Linear Systems > 7.5 Linear Programming

Without graphing, in Exercises 64–67, determine if each system has no solution or infinitely many solutions.

64. \[
\begin{align*}
3x + y & < 9 \\
3x + y & > 9
\end{align*}
\]

65. \[
\begin{align*}
6x - y & \leq 24 \\
6x - y & > 24
\end{align*}
\]

66. \[
\begin{align*}
3x + y & \leq 9 \\
3x + y & \geq 9
\end{align*}
\]

67. \[
\begin{align*}
6x - y & \leq 24 \\
6x - y & \geq 24
\end{align*}
\]

7.5 Linear Programming

What am I Supposed to Learn?
After you have read this section, you should be able to:

1. Write an objective function describing a quantity that must be maximized or minimized.

2. Use inequalities to describe limitations in a situation.

3. Use linear programming to solve problems.

Objective Functions in Linear Programming

1. Write an objective function describing a quantity that must be maximized or minimized.

Many problems involve quantities that must be maximized or minimized. Businesses are interested in maximizing profit. An operation in which bottled water and medical kits are shipped to earthquake survivors needs to maximize the number of survivors helped by this shipment. An objective function is an algebraic expression in two or more variables describing a quantity that must be maximized or minimized.

Example 1 Writing an Objective Function

Bottled water and medical supplies are to be shipped to survivors of an earthquake by plane. Each container of bottled water will serve ten people and each medical kit will aid six people. If \( x \) represents the number of bottles of water to be shipped and \( y \) represents the number of medical kits, write the objective function that describes the number of people who can be helped.
SOLUTION
Because each bottle of water serves ten people and each medical kit aids six people, we have

\[
\text{The number of people helped is} \quad 10 \text{ times the number of bottles of water plus 6 times the number of medical kits.}
\]

Using \(z\) to represent the number of people helped, the objective function is

\[
z = 10x + 6y.
\]

Unlike the functions that we have seen so far, the objective function is an equation in three variables. For a value of \(x\) and a value of \(y\), there is one and only one value of \(z\). Thus, \(z\) is a function of \(x\) and \(y\).

Check Point 1
A company manufactures bookshelves and desks for computers. Let \(x\) represent the number of bookshelves manufactured daily and \(y\) the number of desks manufactured daily. The company's profits are $25 per bookshelf and $55 per desk. Write the objective function that describes the company's total daily profit, \(z\), from \(x\) bookshelves and \(y\) desks. (Check Points 2 through 4 are also related to this situation, so keep track of your answers.)

Constraints in Linear Programming
2 Use inequalities to describe limitations in a situation.

Ideally, the number of earthquake survivors helped in Example 1 should increase without restriction so that every survivor receives water and medical kits. However, the planes that ship these supplies are subject to weight and volume restrictions. In linear programming problems, such restrictions are called constraints. Each constraint is expressed as a linear inequality. The list of constraints forms a system of linear inequalities.

Example 2 Writing a Constraint
Each plane can carry no more than 80,000 pounds. The bottled water weighs 20 pounds per container and each medical kit weighs 10 pounds. Let \(x\) represent the number of bottles of water to be shipped and \(y\) the number of medical kits. Write an inequality that describes this constraint.

SOLUTION
Because each plane can carry no more than 80,000 pounds, we have

\[
The \text{ total weight of the water bottles plus the total weight of the medical kits must be less than or equal to} \quad 80,000 \text{ pounds.}
\]

Each bottle weighs 20 pounds.

Each kit weighs 10 pounds.

The plane's weight constraint is described by the inequality

\[
20x + 10y \leq 80,000.
\]
To maintain high quality, the company in Check Point 1 should not manufacture more than a total of 80 bookshelves and desks per day. Write an inequality that describes this constraint.
Solving Problems with Linear Programming

3 Use linear programming to solve problems.

The problem in the earthquake situation described previously is to maximize the number of survivors who can be helped, subject to each plane’s weight and volume limitations. The process of solving this problem is called linear programming, based on a theorem that was proven during World War II.

Solving a Linear Programming Problem

Let \( z = ax + by \) be an objective function that depends on \( x \) and \( y \). Furthermore, \( z \) is subject to a number of constraints on \( x \) and \( y \). If a maximum or minimum value of \( z \) exists, it can be determined as follows:

1. Graph the system of inequalities representing the constraints.
2. Find the value of the objective function at each corner, or vertex, of the graphed region. The maximum and minimum of the objective function occur at one or more of the corner points.
Example 4 Solving a Linear Programming Problem

Determine how many bottles of water and how many medical kits should be sent on each plane to maximize the number of earthquake survivors who can be helped.

SOLUTION

We must maximize \( z = 10x + 6y \) subject to the following constraints:

\[
\begin{align*}
20x + 10y & \leq 80,000 \\
x + y & \leq 6000.
\end{align*}
\]

Step 1 Graph the system of inequalities representing the constraints. Because \( x \) (the number of bottles of water per plane) and \( y \) (the number of medical kits per plane) must be nonnegative, we need to graph the system of inequalities in quadrant I and its boundary only.

To graph the inequality \( 20x + 10y \leq 80,000 \), we graph the equation \( 20x + 10y = 80,000 \) as a solid blue line (Figure 7.39). Setting the \( x \)-intercept is 4000 and setting the \( y \)-intercept is 8000. Using \((0,0)\) as a test point, the inequality is satisfied, so we shade below the blue line, as shown in yellow in Figure 7.39.

Now we graph \( x + y \leq 6000 \) by first graphing \( x + y = 6000 \) as a solid red line. Setting \( y = 0 \), the \( x \)-intercept is 6000. Setting \( x = 0 \), the \( y \)-intercept is 6000. Using \((0,0)\) as a test point, the inequality is satisfied, so we shade below the red line, as shown using green vertical shading in Figure 7.39.

We use the addition method to find where the lines \( 20x + 10y = 80,000 \) and \( x + y = 6000 \) intersect.

\[
\begin{align*}
20x + 10y & = 80,000 \\
x + y & = 6000
\end{align*}
\]

No change

Multiply by \(-10\)

Add:

\[
\begin{align*}
20x + 10y & = 80,000 \\
-10x - 10y & = -60,000
\end{align*}
\]

\[
\begin{align*}
10x & = 20,000 \\
\hline
x & = 2000
\end{align*}
\]

Back-substituting 2000 for \( x \) in \( z = 10x + 6y \), we find \( y = 4000 \), so the intersection point is \((2000, 4000)\).

Step 2 Find the value of the objective function at each corner of the graphed region. The maximum and minimum of the objective function occur at one or more of the corner points. We must evaluate the objective function, \( z = 10x + 6y \), at the four corners, or vertices, of the region in Figure 7.40.

FIGURE 7.39 The region in quadrant I representing the constraints \( 20x + 10y \leq 80,000 \) and \( x + y \leq 6000 \)

Now we graph \( x + y \leq 6000 \) by first graphing \( x + y = 6000 \) as a solid red line. Setting \( y = 0 \), the \( x \)-intercept is 6000. Setting \( x = 0 \), the \( y \)-intercept is 6000. Using \((0,0)\) as a test point, the inequality is satisfied, so we shade below the red line, as shown using green vertical shading in Figure 7.39.

FIGURE 7.40

FIGURE 7.40 The red and blue line segments are included in the graph.
Corner \((x, y)\) | Objective Function \(z = 10x + 6y\)  
---|---
\((0,0)\) | \(z = 10(0) + 6(0) = 0\)
\((4000,0)\) | \(z = 10(4000) + 6(0) = 40,000\)
\((2000,4000)\) | \(z = 10(2000) + 6(4000) = 44,000 \leftarrow \text{maximum}\)
\((0,6000)\) | \(z = 10(0) + 6(6000) = 36,000\)

Thus, the maximum value of \(z\) is 44,000 and this occurs when \(x = 2000\) and \(y = 4000\). In practical terms, this means that the maximum number of earthquake survivors who can be helped with each plane shipment is 44,000. This can be accomplished by sending 2000 water bottles and 4000 medical kits per plane.
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**Check Point 4**

For the company in Check Points 1–3, how many bookshelves and how many desks should be manufactured per day to obtain a maximum profit? What is the maximum daily profit?

**Concept and Vocabulary Check**

Fill in each blank so that the resulting statement is true.

1. A method for finding the maximum or minimum value of a quantity that is subject to various limitations is called ________________________.

2. An algebraic expression in two or more variables describing a quantity that must be maximized or minimized is called a/an ___________________ function.

3. A system of linear inequalities is used to represent restrictions, or ___________________, on a function that must be maximized or minimized. Using the graph of such a system of inequalities, the maximum and minimum values of the function occur at one or more of the ___________________ points.

**Exercise Set 7.5**

**Practice Exercises**

In Exercises 1–4, find the value of the objective function at each corner of the graphed region. What is the maximum value of the objective function? What is the minimum value of the objective function?

1. Objective Function
   
   \[ z = 5x + 6y \]

   ![Graph](image)

2. Objective Function
   
   \[ z = 3x + 2y \]

   ![Graph](image)

3. Objective Function
   
   \[ z = 40x + 50y \]

   ![Graph](image)
4. Objective Function

\[ z = 30x + 45y \]

In Exercises 5–8, an objective function and a system of linear inequalities representing constraints are given.

a. Graph the system of inequalities representing the constraints.

b. Find the value of the objective function at each corner of the graphed region.

c. Use the values in part (b) to determine the maximum value of the objective function and the values of \( x \) and \( y \) for which the maximum occurs.

5. Objective Function \( z = x + y \) Constraints

\[
\begin{align*}
&x \leq 6 \\
&y \geq 1 \\
&2x - y \geq -1
\end{align*}
\]

6. Objective Function \( z = 3x - 2y \) Constraints

\[
\begin{align*}
&x \geq 1 \\
&x \leq 5 \\
&y \geq 2 \\
&x - y \geq -3
\end{align*}
\]

7. Objective Function \( z = 6x + 10y \) Constraints

\[
\begin{align*}
&x + 2y \leq 12 \\
&x + 2y \leq 20 \\
&x \geq 0 \text{ Quadrant I and} \\
&y \geq 0 \text{ its boundary}
\end{align*}
\]

8. Objective Function \( z = x + 3y \) Constraints

\[
\begin{align*}
&x + y \geq 2 \\
&x \leq 6 \\
&y \leq 5 \\
&x \geq 0 \text{ Quadrant I and} \\
&y \geq 0 \text{ its boundary}
\end{align*}
\]

Practice Plus

Use the directions for Exercises 5–8 to solve Exercises 9–12.

9. Objective Function \( z = 5x - 2y \) Constraints

\[
\begin{align*}
&0 \leq z \leq 5 \\
&0 \leq y \leq 3 \\
&x + y \geq 2
\end{align*}
\]

10. Objective Function \( z = 3x - 2y \) Constraints

\[
\begin{align*}
&1 \leq x \leq 5 \\
&y \geq 2 \\
&x - y \geq -3
\end{align*}
\]
7 Algebra: Graphs, Functions and Linear Systems > 7.5 Linear Programming > Concept and Vocabulary Check

11. Objective Function \( z = 10x + 12y \) Constraints
\[
\begin{align*}
x & \geq 0, \; y \geq 0 \\
x + y & \leq 7 \\
2x + y & \leq 10 \\
2x + 3y & \leq 18
\end{align*}
\]

12. Objective Function \( z = 5x + 6y \) Constraints
\[
\begin{align*}
x & \geq 0, \; y \geq 0 \\
2x + y & \geq 10 \\
x + 2y & \geq 10 \\
x + y & \leq 10
\end{align*}
\]

Application Exercises

13. a. A student earns $10 per hour for tutoring and $7 per hour as a teacher's aide. Let \( z \) = the number of hours each week spent tutoring and \( y \) = the number of hours each week spent as a teacher's aide. Write the objective function that describes total weekly earnings.

b. The student is bound by the following constraints:
   - To have enough time for studies, the student can work no more than 20 hours per week.
   - The tutoring center requires that each tutor spend at least three hours per week tutoring.
   - The tutoring center requires that each tutor spend no more than eight hours per week tutoring.

Write a system of three inequalities that describes these constraints.

c. Graph the system of inequalities in part (b). Use only the first quadrant and its boundary, because \( x \) and \( y \) are nonnegative.

d. Evaluate the objective function for total weekly earnings at each of the four vertices of the graphed region. [The vertices should occur at \( (3, 0) \), \( (8, 0) \), \( (3, 17) \), and \( (8, 12) \).]

e. Complete the missing portions of this statement: The student can earn the maximum amount per week by tutoring for ____ hours per week and working as a teacher's aide for ____ hours per week. The maximum amount that the student can earn each week is $ ____.

14. A television manufacturer makes LCD and plasma televisions. The profit per unit is $125 for the LCD televisions and $200 for the plasma televisions.

a. Let \( x \) = the number of LCD televisions manufactured in a month and \( y \) = the number of plasma televisions manufactured in a month. Write the objective function that describes the total monthly profit.

b. The manufacturer is bound by the following constraints:
   - Equipment in the factory allows for making at most 450 LCD televisions in one month.
   - Equipment in the factory allows for making at most 200 plasma televisions in one month.
   - The cost to the manufacturer per unit is $600 for the LCD televisions and $900 for the plasma televisions. Total monthly costs cannot exceed $360,000.

Write a system of three inequalities that describes these constraints.

c. Graph the system of inequalities in part (b). Use only the first quadrant and its boundary, because \( x \) and \( y \) must both be nonnegative.

d. Evaluate the objective function for total monthly profit at each of the five vertices of the graphed region. [The vertices should occur at \( (0, 0) \), \( (0, 200) \), \( (300, 200) \), \( (450, 100) \), and \( (450, 0) \).]

e. Complete the missing portions of this statement: The television manufacturer will make the greatest profit by manufacturing _____ LCD televisions each month and _____ plasma televisions each month. The maximum monthly profit is $ ____.

15. Food and clothing are shipped to survivors of a natural disaster. Each carton of food will feed 12 people, while each carton of clothing will help 5 people. Each 20-cubic-foot box of food weighs 50 pounds and each 10-cubic-foot box of clothing weighs 20 pounds. The commercial carriers transporting food and clothing are bound by the following constraints:
   - The total weight per carrier cannot exceed 19,000 pounds.
   - The total volume must be no more than 8000 cubic feet.

How many cartons of food and clothing should be sent with each plane shipment to maximize the number of people who can be helped?

16. You are about to take a test that contains computation problems worth 6 points each and word problems worth 10 points each. You can do a computation problem in 2 minutes and a word problem in 4 minutes. You have 40 minutes to take the test and may answer no more than 12 problems. Assuming you answer all the problems attempted correctly, how many of each type of problem must you answer to maximize your score? What is the maximum score?

17. A theater is presenting a program on drinking and driving for students and their parents. The proceeds will be donated to a local alcohol information center. Admission is $2 for parents and $1 for students. However, the situation has two constraints: The theater can hold no more than 150 people and every two parents
must bring at least one student. How many parents and students should attend to raise the maximum amount of money?

18. On June 24, 1948, the former Soviet Union blocked all land and water routes through East Germany to Berlin. A gigantic airlift was organized using American and British planes to bring food, clothing, and other supplies to the more than 2 million people in West Berlin. The cargo capacity was 30,000 cubic feet for an American plane and 20,000 cubic feet for a British plane. To break the Soviet blockade, the Western Allies had to maximize cargo capacity, but were subject to the following restrictions:

- No more than 44 planes could be used.
- The larger American planes required 16 personnel per flight, double that of the requirement for the British planes. The total number of personnel available could not exceed 512.
- The cost of an American flight was $9000 and the cost of a British flight was $5000. Total weekly costs could not exceed $300,000.

Find the number of American and British planes that were used to maximize cargo capacity.
7 Algebra: Graphs, Functions and Linear Systems  >  7.6 Modeling Data: Exponential, Logarithmic, and Quadratic Functions

Writing in Mathematics

19. What kinds of problems are solved using the linear programming method?

20. What is an objective function in a linear programming problem?

21. What is a constraint in a linear programming problem? How is a constraint represented?

22. In your own words, describe how to solve a linear programming problem.

23. Describe a situation in your life in which you would like to maximize something, but you are limited by at least two constraints. Can linear programming be used in this situation? Explain your answer.

Critical Thinking Exercises

Make Sense? In Exercises 24–27, determine whether each statement makes sense or does not make sense, and explain your reasoning.

24. In order to solve a linear programming problem, I use the graph representing the constraints and the graph of the objective function.

25. I use the coordinates of each vertex from my graph representing the constraints to find the values that maximize or minimize an objective function.

26. I need to be able to graph systems of linear inequalities in order to solve linear programming problems.

27. An important application of linear programming for businesses involves maximizing profit.

28. Suppose that you inherit $10,000. The will states how you must invest the money. Some (or all) of the money must be invested in stocks and bonds. The requirements are that at least $3000 be invested in bonds, with expected returns of $0.08 per dollar, and at least $2000 be invested in stocks, with expected returns of $0.12 per dollar. Because the stocks are medium risk, the final stipulation requires that the investment in bonds should never be less than the investment in stocks. How should the money be invested so as to maximize your expected returns?

Group Exercises

29. Group members should choose a particular field of interest. Research how linear programming is used to solve problems in that field. If possible, investigate the solution of a specific practical problem. Present a report on your findings, including the contributions of George Dantzig, Narendra Karmarkar, and L. G. Khachion to linear programming.

30. Members of the group should interview a business executive who is in charge of deciding the product mix for a business. How are production policy decisions made? Are other methods used in conjunction with linear programming? What are these methods? What sort of academic background, particularly in mathematics, does this executive have? Present a group report addressing these questions, emphasizing the role of linear programming for the business.

7.6 Modeling Data: Exponential, Logarithmic, and Quadratic Functions

What am I Supposed to Learn?

After you have read this section, you should be able to:

1 Graph exponential functions.
2 Use exponential models.
3 Graph logarithmic functions.
4 Use logarithmic models.
5 Graph quadratic functions.
6 Use quadratic models.
7 Determine an appropriate function for modeling data.

IS THERE A RELATIONSHIP BETWEEN LITERACY AND child mortality? As the percentage of adult females who are literate increases, does the mortality of children under five decrease? Figure 7.41, based on data from the United Nations, indicates that this is, indeed, the case. Each point in the figure represents one country.
Data presented in a visual form as a set of points are called a scatter plot. Also shown in Figure 7.41 is a line that passes through or near the points. The line that best fits the data points in a scatter plot is called a regression line. We can use the line’s slope and y-intercept to obtain a linear model for under-five mortality, $y$, per thousand, as a function of the percentage of literate adult females, $x$. The model is given at the top of the next page.